Cornerstones

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Geometric Function Theory

*Explorations in Complex Analysis*

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To the memory of Don Spencer
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Preface

Complex analysis is a rich and textured subject. It is quite old, and its history is broad and deep. Yet the basic graduate course in complex variables has become rather cut and dried. The choice of topics, the order of the topics, and the overall flavor of the presentation are strongly influenced by the need to prepare students for the qualifying exams. The qual-level course is designed to serve a limited purpose, and it does that but little more.

Basic complex analysis is startling for its elegance and clarity. One progresses very rapidly from the basics of the Cauchy theory to profound results such as the fundamental theorem of algebra and the Riemann mapping theorem. Many a student is left, at the end of the course, yearning for more—to be advanced to a level where he or she could consider research questions, indeed the possibility of writing a thesis in the subject.

Yet there are few places to turn in such a quest. The book [BEG] of Berenstein and Gay and the book [CON] of Conway give particular takes on some of the more advanced material in the subject. Some of the older books, such as [FUKS], [GOL], [HIL], [MAR] treat topics not usually found in the basic texts. But it seems that there is a need for a book that will open the student’s eyes to what this subject has to offer, and give him or her a taste of some of the areas of current research. This is meant to be such a book.

Our prejudice in the subject is geometric, but this does not prevent us from exploring byways that come from analysis, algebra, and other parts of mathematics. Thus, on the one hand, we treat invariant geometry, the Bergman metric, the automorphism groups of domains, and the boundary regularity of conformal mappings. On the other hand, we also explore the Hilbert transform, the Laplacian, the corona problem, harmonic measure, the inhomogenous Cauchy–Riemann equations, and sheaf theory.

The aim of the book is to expose the student to mathematics as it is practiced: as a synthesis of many different areas, exhibiting particular flavors and features that arise from that synthesis. The student who reads this book should be inspired to go further in the subject, to begin to explore the primary literature, and (one hopes) to think about his or her own research problems.
One of the rewards and pleasures that the student will find in reading this book is the rich interactions that are displayed among the various topics. For example, harmonic measure is used to prove a sharp version of the Lindelöf principle. It is also used to establish the three lines (viz. three circles) theorem of Hadamard. This in turn is used to prove (in another chapter) the Riesz–Thorin theorem. In another venue, the Riesz–Thorin theorem is used to prove the $L^p$-boundedness of the Hilbert transform. The Laplacian is reviewed and used as a device to introduce the Green’s function. The Green’s function is used, of course, to derive the Poisson kernel. But it is also used to develop the Bergman kernel. The Bergman kernel is used to study boundary regularity of conformal mappings. It is also used in the study of automorphism groups, and to prove various uniqueness theorems for conformal mappings. The Bergman metric interacts with and arises alongside consideration of other conformal metrics, and leads to Ahlfors’s version of the Schwarz lemma.

The Poisson kernel is used to study the boundary behavior of harmonic and holomorphic functions. The Dirichlet problem for the Laplacian is used to give a nonstandard proof of the Riemann mapping theorem. The Green’s function is used to prove a canonical representation of multiply connected regions on slit-domains. The Ahlfors map gives a new view of uniformization as introduced by these topics.

The Green’s function and Stokes’s theorem lead to a solution of the inhomogeneous Cauchy–Riemann equations, which are in turn used to make new constructions in function theory. The Green’s function and the Hilbert transform, together with our solution of the inhomogeneous Cauchy–Riemann equations and the F. and M. Riesz theorem, are used to derive a proof of the corona theorem. Duality properties of Hardy spaces (covered in an earlier chapter) are exploited along the way.

Our study of the Ahlfors map uses Banach algebra properties of $H^\infty$ that we developed in our study of the corona theorem. It also gives a reprise of the Green’s function and harmonic measure. The Green’s function and Green’s theorem are used extensively in the proof of the corona theorem and also in our development of the uniformization theorem. Nontrivial ideas from functional analysis arise in our study of the Riesz–Thorin theorem, the Hilbert transform, the summation of Fourier series, the corona theorem, and the Ahlfors map.

We provide a discussion of the statement, concept, and proof of Köbe’s uniformization theorem, together with various planar variants. Thus we examine uniformization from many different points of view. When we treat automorphism groups, uniformization provides a powerful tool.

Algebra is encountered in various guises throughout the book. Certainly it plays a role in the group-theoretic aspects of automorphisms. It occurs again in our treatment of Banach algebra techniques. And it plays a decisive role in the study of sheaves. Sheaf theory gives the student a new way to view the Weierstrass and Mittag–Leffler theorems, as well as questions of analytic continuation. Thus this text shows the students many different aspects of complex analysis, and how they interact with each other.
With this book, the student and the advanced worker too are introduced to a rich tapestry of function theory as it interacts with other parts of mathematics. There is hardly any other analysis text that offers such a variety and synthesis of mathematical topics.

Part of my own training is to think of the subject of complex analysis as a foil, or perhaps as a gadfly. Many an interesting problem of geometric analysis is very naturally formulated in the language of complex function theory; but then it is best solved by stripping away the complex analysis and applying tools of geometry, or partial differential equations, or harmonic analysis. That is the point of view that we shall, at least in part, take in the present book. Complex analysis will be our touchstone, but it will be the entrée to many another byway of mathematics—from the Cauchy–Riemann equations to interpolation of linear operators to the study of invariant metrics. It is our view that this is a productive and rewarding way to practice mathematics, and we would like to teach it to a new generation.

It is a pleasure to thank my editor, Ann Kostant, for encouraging me to write this book, and for making the process as smooth and carefree as possible. She enlisted strong and insightful reviewers to help me craft this book into a more precise and useful tool. I thank Elizabeth Loew for a marvelous job of editing and $\TeX$ typesetting. I look forward to comments and criticisms from the readership, and hope to make future editions more accurate and therefore more useful.

St. Louis, Missouri and Berkeley, California  

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Part I

Classical Function Theory
Invariant Geometry

Genesis and Development

The idea of using invariant geometry to study complex function theory has its foundation in the ideas of Poincaré. Certainly he is credited with the creation of a conformally invariant metric on the unit disk $D$. The uniformization theorem (covered later in this book) may be used to transfer the metric to other planar domains. Later on, Stefan Bergman found a way to define invariant metrics on virtually any domain in any complex manifold. We shall explore his ideas further on in the book.

The geometric approach provides a new way to view the subject of complex variables. It is the source of tantalizing new questions. But it also provides a vast array of powerful new weapons to use on traditional problems. Any number of problems about mappings and conformality—just as an instance—are rendered transparent by way of geometric language.

An appreciation of the concepts in this chapter requires understanding of the idea of a Riemannian metric. Our presentation, however, is self-contained. The reader may learn what such a metric is by doing, that is, by studying this chapter. Even the student who is new to geometric language will find that the material in the ensuing pages will introduce him to a new way to approach function theory.

1.1 Conformality and Invariance

Conformal mappings are characterized by the fact that they infinitesimally (i) preserve angles, and (ii) preserve length (up to a scalar factor). It is worthwhile to picture the matter in the following manner: Let $f$ be holomorphic on the open set $U \subset \mathbb{C}$. Fix a point $P \in U$. Write $f = u + iv$ as usual. Thus we may write the mapping $f$ as $(x, y) \mapsto (u, v)$. Then the (real) Jacobian matrix of the mapping is
\[ J(P) = \begin{pmatrix} u_x(P) & u_y(P) \\ v_x(P) & v_y(P) \end{pmatrix}, \]

where subscripts denote derivatives. We may use the Cauchy–Riemann equations to rewrite this matrix as

\[ J(P) = \begin{pmatrix} u_x(P) & u_y(P) \\ -u_y(P) & u_x(P) \end{pmatrix}. \]

Factoring out a numerical coefficient, we finally write this two-dimensional derivative as

\[ J(P) = \sqrt{u_x(P)^2 + u_y(P)^2} \cdot \begin{pmatrix} u_x(P) \sqrt{u_x(P)^2 + u_y(P)^2} \\ -u_y(P) \sqrt{u_x(P)^2 + u_y(P)^2} \end{pmatrix} \]

\[ \equiv h(P) \cdot J(P). \]

The matrix \( J(P) \) is of course a special orthogonal matrix (i.e., its rows form an orthonormal basis of \( \mathbb{R}^2 \), and it is oriented positively). Thus we see that the derivative of our mapping is a rotation \( J(P) \) (which preserves angles) followed by a positive “stretching factor” \( h(P) \) (which also preserves angles).

It would be incorrect to infer from these considerations that a conformal map therefore preserves Euclidean angles, or Euclidean length, in any global sense. Such a mapping would perforce be linear and in fact special orthogonal; and thus complex function theory would be reduced to a triviality. The mapping of the disk \( D(0,2) \) given by

\[ \phi : z \mapsto (z + 4)^4, \]

illustrated in Figure 1.1, exhibits rather dramatically the failure of Euclidean isometry. The mapping is one-to-one, onto its image, yet \( \phi(0) = 256, \phi(1) = 625 \), and hence

\[ 1 - 0 = 1 \neq 369 = \phi(1) - \phi(0). \]

And yet it is obviously desirable to have a notion of distance that is preserved under holomorphic mappings. Part of Klein’s *Erlangen program* is to understand geometric objects according to the groups that act on them. Such an understanding is of course facilitated when the group is actually a group of isometries.

It was H. Poincaré who first found a way to carry out this idea when the domain in question is the unit disk \( D \). To understand his thinking, let us recall that the collection of all conformal maps of the unit disk \( D = D(0,1) \) can be described explicitly; it consists of
1.1 Conformality and Invariance

Fig. 1.1. The failure of Euclidean isometry under conformal mappings.

(i) All rotations $\rho_\lambda : z \mapsto e^{i\lambda} \cdot z$, $0 \leq \lambda < 2\pi$;
(ii) All Möbius transformations $\varphi_a : z \mapsto [z - a]/[1 - \bar{a}z]$, $a \in \mathbb{C}$, $|a| < 1$;
(iii) All compositions of mappings of type (i) and (ii).

We will use Riemann’s paradigm for a metric, that is, we shall specify the length of a (tangent) vector at each point of the disk $D$. If the point is $P$ and the vector is $\mathbf{v}$, then let us denote this length by $|\mathbf{v}|_P$.

Our goal now is to “discover” the Poincaré metric by way of a sequence of calculations. We must begin somewhere, so let us declare that the length of the vector $\mathbf{e} \equiv (1, 0) \equiv 1 + 0i$ at the origin is 1. Thus $|\mathbf{e}|_0 = 1$. Now if $\phi$ is a conformal self-map of the disk, then invariance means that

$$|\mathbf{v}|_P = |\phi_*(P)\mathbf{v}|_{\phi(P)}, \quad (1.1.1)$$

where $\phi_*(P)\mathbf{v}$ is defined to equal $\phi'(P) \cdot \mathbf{v}$. Here $\phi_*(P)\mathbf{v}$ is called the “pushforward by $\phi$” of the vector $\mathbf{v}$.

Now let us apply equation (1.1.1) to the information that we have about the length of the unit vector $\mathbf{e}$ at the origin. Let $\phi(z) = e^{i\lambda} \cdot z$. Then we see that

$$1 = |\mathbf{e}|_0 = |\phi_*(0)\mathbf{e}|_{\phi(0)} = |e^{i\lambda} \cdot \mathbf{e}|_0 = |e^{i\lambda}|_0. \quad (1.1.2)$$

We conclude that the length of the vector $e^{i\lambda}$ at the origin is 1. So all Euclidean unit vectors based at the origin have length 1 in our new invariant metric.

Now let $\psi(z)$ be the Möbius transformation

$$\psi(z) = \frac{z + a}{1 + \bar{a}z},$$

with $a$ a complex number of modulus less than 1. Let $\mathbf{v} = e^{i\lambda}$ be a unit vector at the origin. Then $\psi'(0) = 1 - |a|^2$ and equation (1.1.1) tells us that
We conclude that
\[ |v|_a = \frac{1}{1 - |a|^2}. \]

Of course our new metric will respect scalar multiplication, so we may apply the preceding calculation to vectors of any length. If we let \( ||v|| \) denote the Euclidean length of a vector \( v \), then we may summarize all our calculations as follows:

Let \( P \) be a point of the unit disk \( D \) and let \( v \) be any vector based at that point. Then
\[ |v|_P = \frac{||v||}{1 - |P|^2}. \]

We call this new metric the Poincaré metric.

We see in particular that, as \( P \) gets ever nearer to the boundary of \( D \), the new metric length of \( v \) becomes greater and greater.

It is well worth spending some time interpreting this new metric and its implications for complex analysis. Let \( \gamma : [0, 1] \rightarrow D \) be a continuously differentiable curve. We define the length \( \ell \) of \( \gamma \) in the Poincaré metric to be
\[ \ell(\gamma) = \int_0^1 |\gamma'(t)|_{\gamma(t)} \, dt. \]

Observe that \( \gamma'(t) \) is a vector located at \( \gamma(t) \), so the definition makes good sense. Of course we can consider curves parametrized over any interval, and the definition of length is independent (by the change of variables formula of calculus) of the choice of parametrization. We define the length of a piecewise continuously differentiable curve to be the sum of the lengths of its continuously differentiable pieces.

**Example 1.1.1.** Let \( \epsilon > 0 \). Consider the curve \( \gamma(t) = (1 - \epsilon)t, 0 \leq t \leq 1 \). Then, according to the definition,
\[ \ell(\gamma) = \int_0^1 |\gamma'(t)|_{\gamma(t)} \, dt = \int_0^1 \frac{(1 - \epsilon)}{1 - |\gamma(t)|^2} \, dt \]
\[ = \int_0^1 \frac{(1 - \epsilon)}{1 - [(1 - \epsilon)t]^2} \, dt = \frac{1}{2} \log \left( \frac{2 - \epsilon}{\epsilon} \right). \]

We see immediately that, as \( \epsilon \rightarrow 0^+ \), the expression \( \ell(\gamma) \) tends to \( +\infty \). Thus, in some sense, the distance from 0 to the boundary (at least along the given linear path) is infinite.

We define the Poincaré distance \( d(P, Q) \) between two points \( P, Q \in D \) to be the infimum of the Poincaré lengths of all piecewise continuously differentiable curves connecting \( P \) to \( Q \).
Proposition 1.1.2. Let $P \in D$. Then the Poincaré distance of 0 to $P$ is equal to
\[
d(0, P) = \frac{1}{2} \cdot \log \frac{1 + |P|}{1 - |P|}.
\]

Proof. By rotational invariance, we may as well suppose that $P$ is real and positive, so $P = (1 - \epsilon) + i0 \equiv (1 - \epsilon, 0)$. It is an exercise for the reader to see that there is no loss of generality to consider only curves of the form $\gamma(t) = (t, g(t)), 0 \leq t \leq 1 - \epsilon$. Then
\[
\ell(g) = \int_0^{1-\epsilon} \frac{\|\gamma'(t)\|}{1 - |\gamma(t)|^2} dt = \int_0^{1-\epsilon} \frac{\sqrt{1 + |g'(t)|^2}}{1 - t^2 - |g(t)|^2} dt
\]
\[
\geq \int_0^{1-\epsilon} \frac{1}{1 - t^2} dt = \frac{1}{2} \cdot \log \left( \frac{2 - \epsilon}{\epsilon} \right)
\]
\[
= \frac{1}{2} \cdot \log \left( \frac{1 + |P|}{1 - |P|} \right).
\]

Thus we see explicitly that $\mu(t) = (t, 0)$ is the shortest curve from 0 to $P$, and the distance is as we claimed.

Combining the result of this proposition with the preceding example, we see that any curve that starts at the origin and runs out to the boundary of the disk will have infinite length. Thus the boundary is infinitely far away.

Exercise for the Reader: Prove that if $P$ is any point of the disk and $r > 0$, then the metric disk $\beta(P, r) \equiv \{z \in D : d(z, P) < r\}$ is actually a Euclidean disk. What are its Euclidean center and radius (expressed in terms of $r$ and $P$)? Show that the disk $\beta(P, r)$ is relatively compact (i.e., has compact closure) in the disk. It follows that any Cauchy sequence in the Poincaré metric has a limit point in the disk. Thus the Poincaré metric turns $D$ into a complete metric space.

1.2 Bergman’s Construction

Stefan Bergman created a device for equipping virtually any planar domain with an invariant metric that is analogous to the Poincaré metric on the disk. Some tracts call this new metric the Poincaré–Bergman metric, though it is

Later on, in Section 4.6, we shall treat the uniformization theorem of Köbe. It gives a means for transferring the Poincaré metric from the disk to virtually any planar domain.
more commonly called just the Bergman metric. In order to construct the Bergman metric we must first construct the Bergman kernel. For that we need just a little Hilbert space theory (see [RUD2], for example).

A domain in \( \mathbb{C} \) is a connected open set. Fix a domain \( \Omega \subseteq \mathbb{C} \), and define

\[
A^2(\Omega) = \left\{ f \text{ holomorphic on } \Omega : \int_{\Omega} |f(z)|^2 dA(z) < \infty \right\} \subseteq L^2(\Omega).
\]

Here \( dA \) is an ordinary two-dimensional area measure. Of course \( A^2(\Omega) \) is a complex linear space, called the Bergman space. The norm on \( A^2(\Omega) \) is given by

\[
\|f\|_{A^2(\Omega)} = \left( \int_{\Omega} |f(z)|^2 dA(z) \right)^{1/2}.
\]

We define an inner product on \( A^2(\Omega) \) by

\[
\langle f, g \rangle = \int_{\Omega} f(z)g(z) dA(z).
\]

The next technical lemma will be the key to our analysis of the space \( A^2 \).

**Lemma 1.2.1.** Let \( K \subseteq \Omega \) be compact. There is a constant \( C_K > 0 \), depending on \( K \), such that

\[
\sup_{z \in K} |f(z)| \leq C_K \|f\|_{A^2(\Omega)}, \quad \text{all } f \in A^2(\Omega).
\]

**Proof.** Since \( K \) is compact, there is an \( r(K) = r > 0 \) so that, for any \( z \in K, D(z, r) \subseteq \Omega \). Therefore, for each \( z \in K \) and \( f \in A^2(\Omega) \), we may use the mean value property of harmonic functions to see that

\[
|f(z)| = \frac{1}{A(B(z, r))} \left| \int_{B(z, r)} f(t) dA(t) \right| \leq \frac{1}{A(B(z, r))} \cdot \int_C \chi_{B(z, r)}(t) \cdot |f(t)| dA(t),
\]

where

\[
\chi_{B(z, r)}(t) = \begin{cases} 1 & \text{if } t \in B(z, r) \\ 0 & \text{if } t \notin B(z, r). \end{cases}
\]

We apply the Cauchy–Schwarz inequality from integration theory (see [RUD2]) to the last expression to find that it is less than or equal to

\[
\left( \frac{1}{A(B(z, r))} \right) \cdot \int_C |\chi_{B(z, r)}(t)|^2 dA^{1/2} \cdot \int_C |f(t)|^2 dA^{1/2} = (A(B(z, r)))^{-1/2} \|f\|_{A^2(\Omega, B(z, r))} = \frac{1}{\sqrt{\pi r}} \|f\|_{A^2(\Omega)} \equiv C_K \|f\|_{A^2(\Omega)}.
\]

\[\square\]
Proposition 1.2.2. The space $A^2(\Omega)$ is complete.

Proof. Let $\{f_j\}$ be a Cauchy sequence in $A^2$. Then the sequence is Cauchy in $L^2(\Omega)$, and the completeness of $L^2$ (see [RUD2]) then tells us that there is a limit function $f$. So $f_j \to f$ in the $L^2$ topology. Now the lemma tells us that in fact the convergence is taking place uniformly on compact sets. So $f \in A^2(\Omega)$. That completes the argument. \qed

Corollary 1.2.3. The space $A^2(\Omega)$ is a Hilbert space.

With a little extra effort (see [GRK1], [RUD3]), it can be shown that $A^2(\Omega)$ is in fact a separable Hilbert space.

Now our point of view is to find a method for representing certain linear functionals. The key fact is this:

Lemma 1.2.4. For each fixed $z \in \Omega$, the functional

$$\Phi_z : f \mapsto f(z), \ f \in A^2(\Omega)$$

is a continuous linear functional on $A^2(\Omega)$.

Proof. This is immediate from Lemma 1.2.1 if we take $K$ to be the singleton $\{z\}$. \qed

We may now apply the Riesz representation theorem (see [RUD2]) to see that there is an element $k_z \in A^2(\Omega)$ such that the linear functional $\Phi_z$ is represented by inner product with $k_z$: if $f \in A^2(\Omega)$ then, for all $z \in \Omega$, we have

$$f(z) = \langle f, k_z \rangle. \quad (1.2.1)$$

Definition 1.2.5. The Bergman kernel is the function $K(z, \zeta) = k_z(\zeta)$, $z, \zeta \in \Omega$. It has the reproducing property

$$f(z) = \int K(z, \zeta)f(\zeta)\,dA(\zeta), \ \forall f \in A^2(\Omega). \quad (1.2.2)$$

Notice that (1.2.2) is just a restatement of (1.2.1).

Proposition 1.2.6. The Bergman kernel $K(z, \zeta)$ is conjugate symmetric: $K(z, \zeta) = \overline{K(\zeta, z)}$.

Proof. By its very definition, $\overline{K(\zeta, \cdot)} \in A^2(\Omega)$ for each fixed $\zeta$. Therefore the reproducing property of the Bergman kernel gives

$$\int_\Omega K(z, t)\overline{K(\zeta, t)}\,dA(t) = \overline{K(\zeta, z)}. \quad (1.2.3)$$

On the other hand,

$$\int_\Omega K(z, t)\overline{K(\zeta, t)}\,dA(t) = \overline{\int K(\zeta, t)\overline{K(z, t)}\,dA(t)}$$

$$= \overline{K(z, \zeta)} = K(z, \zeta). \quad \Box$$